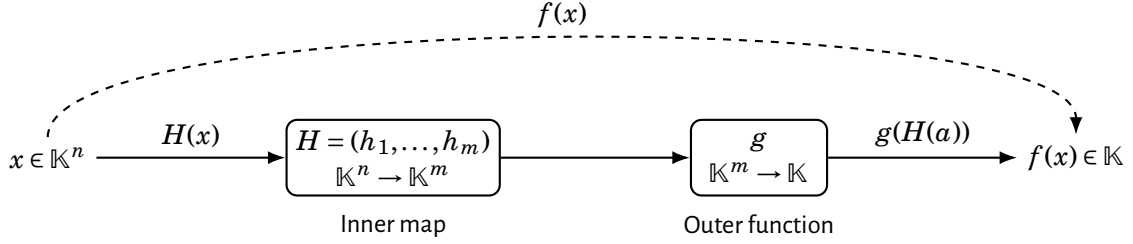


M2 Internship Proposal:

Toward Faster Algorithms for Decomposable Polynomial Systems



Supervision.

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Context and motivation. Decomposable polynomials, those that can be expressed as compositions of lower-degree polynomials, form a classical and fundamental topic in computer algebra, with numerous applications. These include cryptography (see, e.g., [11, 6, 5, 3, 9]), control theory and system identification [4], as well as the analysis and reduction of differential equations (cf. [10] and references therein). For instance, Multivariate Public-Key Cryptosystems (MPKCs) rely on the hardness of solving systems of multivariate polynomial equations over finite fields. A typical public map is of the form

$$F = S \circ C \circ T,$$

where S, T are secret affine transformations and C is a central map of special algebraic form.

This internship aims to develop theoretical and algorithmic foundations for exploiting decomposable structures in multivariate polynomial systems, combining symbolic and numerical techniques.

State of the art. Let f be a polynomial in $\mathbb{K}[x_1, \dots, x_n]$ for a field $\mathbb{K} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. The *Polynomial Decomposition Problem* (PDP), known to be NP-hard [2, 12], asks whether there exist polynomials g and h_i such that

$$f(x_1, \dots, x_n) = g(h_1(x_1, \dots, x_n), \dots, h_m(x_1, \dots, x_n)),$$

and, if so, how to compute such a decomposition. The univariate case is completely solved: polynomial-time algorithms based on factorization in composition algebras are known [8, 1] and implemented in major computer algebra systems (e.g., `compo1y` in Maple). In contrast, multivariate algorithms are only known for homogeneous polynomials of the same degree [6], leaving the general case open. These algorithms typically rely on Gröbner basis computations over various fields. Another common approach is to divide the problem into two steps: (1) computing candidate inner polynomials $h_1, \dots, h_m$, and (2) reconstructing the outer polynomial g . Recent work by T. X. Vu [13] provides a solution to the second step, while the first remains open in the general setting.

Another central problem in computational algebraic geometry and computer algebra is *Polynomial System Solving* (PSS). Such systems arise in many areas, including computer algebra, robotics, geometric modeling, signal processing, cryptology, and molecular biology, and are generally NP-hard [7]. A key bottleneck in symbolic computation is the *intermediate expression swell*, where algebraic data grow combinatorially.

A promising way to mitigate this is by exploiting hidden algebraic structures, such as decomposability, to reduce the computational complexity of PSS techniques such as homotopy continuation methods. Broadly speaking, a homotopy between a target system $F(x) = 0$ to be solved, and a cleverly chosen, easy-to-solve source system $F_0(x) = 0$, is a map $(t, x) \mapsto tF(x) + (1 - t)F_0(x)$ that continuously deforms the known roots of F_0 (at $t = 0$) into those of F (at $t = 1$). Several flavors of it will be investigated in this internship, namely *symbolic*, *numerical* and *hybrid symbolic-numeric* homotopy continuation methods. While symbolic methods guarantee exactness, numerical approaches are approximate but often more scalable and parallelizable. Combining the two yields *hybrid symbolic-numeric algorithms*, which aim to balance precision and efficiency by replacing exact representations with guaranteed enclosures. Adapting such techniques to decomposable systems remains largely unexplored in the context of PSS, precisely the gap this internship aims to bridge.

Objectives of the internship. The goal is to design faster algorithms for problems involving decomposable polynomials and to demonstrate their effectiveness on key applications. The intern will contribute to developing the **theoretical and algorithmic foundations** for **detecting, computing, and exploiting** decompositions of multivariate polynomial maps, with a particular focus on **hybrid symbolic-numeric** methods.

Required skills. A background in algebra, algorithms, and/or numerical analysis, as well as basic programming skills (in Maple, SageMath, Julia, or Python), would be useful but not mandatory. An interest in symbolic-numeric computation and computer algebra is expected.

Practical information.

- Duration: 4-6 months (Spring 2027)
- Location: Université de Lille, CRISTAL (CFHP team)
- Possible continuation: PhD opportunity in Computer Algebra

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